Extracts from

Action Research using Cuisenaire Rods to Develop the Concept of Equivalence with Year 1 Children

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Abstract

This research thesis is based on action research which I undertook with a group of Year 1 children. I used Gattegno’s exercises in his book ‘Mathematics with numbers in colour: Book 1’ to form the basis for every mathematics lesson over the course of an academic year. The children used Cuisenaire rods as the main manipulative. I wanted to see whether this model of teaching and learning mathematics would have a positive impact on their conceptual understanding of equivalence and help them overcome difficulties with the meaning of the equals sign. My primary source of evidence was videoed observations at the midpoint and end of the year, and this data was triangulated using a research diary and children’s work in books.

My results showed that the rods can promote an awareness of equivalence through the process of selecting and moving them. The rods gave the children a model for visualising and understanding abstract concepts such as the deeper structures and rules of equivalence. When the children were given opportunities to “free write” what they had learnt with the rods, with no constraints on what or how to record, they demonstrated they were capable of mastery of a much deeper understanding of what the equals sign means and wrote complex equations of equivalence involving brackets, all four operations and fractions as an operator. Children’s knowledge of the mastery of deeper structures when using the rods can become really powerful when they can move fluently between different representations of equivalence; moving from manipulation of the rods to writing abstract equations using the algebra of letter names for the rods and their composition.

Declaration

I hereby declare that this dissertation is the result of my independent investigation, except where I have indicated my indebtedness to other sources.
I hereby declare that this dissertation has not been submitted or accepted in substance for any other degree, nor is it being submitted currently for any other degree.

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Chapter 1: Introduction

In this chapter I present the context of my research and the relevancy of the topic. I present and justify my research questions and set out the aims and objectives for my research and what I hoped to achieve.

1.1 Context

I am a Maths Specialist (M.A.S.T.) teacher within a very small rural English Primary School with mixed age classes. The school does not have a diverse socio-economic group of children with no EAL (English as an additional language) children and currently only one pupil premium child who was not in the research group. Therefore this school may not be like other schools due to its size and demographics. I teach mathematics to a small group of Year 1 children – the group started in September 2015 as nine children, but two more joined the class halfway through the academic year; however they are not included in the research. I hope that the impact of this research will be a deeper understanding of the mathematical ideas related to equivalence and the use and meaning of the equals sign (=).

1.2 Area of Focus

Following the advice of Wilson and Sutchbury (in Wilson, 2009), I decided to keep my research within a narrow boundary with a clearly manageable focus within the context of my role in school. I have been involved in a project to teach algebra to Year 1 children using Cuisenaire rods (also hereafter, just called rods) since February 2015, starting with an initial trial of materials, which I then adopted from September 2015 as my main resource for teaching and learning in every mathematics lesson. The algebra project includes other schools (three of which I have worked closely with) who are also adopting the Early Algebra approach and the project is led by Doctor Ian Benson, facilitator of Sociality Mathematics, a CPD (Continuing
Professional Development) Network. As I was already involved in this project, it was a natural focus for my research.

The algebra project involves using Gattegno’s exercises in his book ‘Mathematics with numbers in colour: Book 1’ – hereafter, also called Gattegno’s Textbook 1 (Gattegno, 1963) to form the basis for every mathematics lesson. The exercises start with qualitative work which establishes relationships between the rods in terms of equivalent length, and shows that rods can be transformed or swapped around and still have the same equivalence. The rods are given letters to denote them so that equations can be written using algebra to demonstrate these relationships between the rods. Brackets, fractions and multiples are all explored before the rods are attached any numerical value and then they are used to study the numbers up to 20 and to further explore fractions and the operations of addition, difference, multiplication and division.

I was very interested in how this algebra project might develop mathematical understanding of key concepts for the children who were exposed to this way of working, and what the benefits could be to this approach. As there are many concepts being developed I have chosen to narrow my focus further to concentrate on the concept of equivalence and the children’s understanding of the equals sign as this can have many interpretations for children, such as “write the answer now”. It was a relevant area to research because of the work that Sociality Mathematics was already doing in this area.

1.3 Research Title

Action Research using Cuisenaire Rods to Develop the Concept of Equivalence with Year 1 Children
1.4 Research Questions

1. How can Cuisenaire rods be used to promote an awareness of equivalence and help overcome difficulties with the use of the equals sign?
2. How can “free writing” using letter codes to name Cuisenaire rods, promote an awareness of equivalence?
3. How can working with different media, each corresponding to different representations of equivalence, help teachers and learners move fluently between representations of equivalence?

The overall aim of my research was to reflect on the use of Cuisenaire rods to teach equivalence. The objectives were to explore the children’s understanding of equivalence and their use and understanding of the equals sign, to reflect on Gattegno’s model of mathematics teaching in relation to equivalence, and to discuss whether this would have a positive impact on conceptual understanding or not. I hoped to achieve a positive impact on the children’s understanding of equivalence from this approach to learning mathematics.

1.5 Summary

I have shown the relevance and context of my research and presented the research questions, aims and objectives, explaining what I hoped to achieve. Next I will conduct a thorough and critical Literature review, before discussing my research design and methodology. Then I will analyse and present my findings in a critical and evaluative manner, before drawing conclusions.
Chapter 4: Findings and Discussion

In this chapter I present and discuss my findings relating to each of the research questions, with reference to each of the data collection methods and relating back to the deductive framework set out in Chapter 2.

4.1 Research Question 1

*How can Cuisenaire rods be used to promote an awareness of equivalence and help overcome difficulties with the use of the equals sign?*

4.1.1 Video observations

When comparing the videos taken in February 2016 and then in July 2016 using the same missing letter questions about equivalence (see Appendix A for questions), all of the children showed some improved understanding of the equals sign. The children (using pseudonyms) who showed the most improvement were James (least able) and Bethany (who came out of Reception above expected). From my coding, I saw that one of the things that these two children were doing more often now was realising their own mistakes when working with the rods and self correcting, particularly in the case of Bethany. For example:
Bethany (July 2016 video): (Notices the yellow is too short and swaps the yellow for a black which is too long. Then she swaps the black for a dark green).

Bethany (July 2016 video): (Rubs the g out and writes d – spotting her confusion between dark green and light green).

The other two children: Luke (average ability) and Kerry (most able), had the same amount of self correction in both videos, with Kerry, as I might have expected, making the least mistakes. Referring back to my deductive framework, I suggest that this awareness of having made an error when using the equals sign, is leading to ‘know-how’ and knowledge (Gattegno, 1971; Coles, 2015). This perception of the mistake in an equivalence equation is occurring through action (Gattegno, 1974, 1987), highlighting the importance of the manipulation of the rods in order to gain understanding, which is Goutard’s (1964) empirical mode of thought. This will then lead to systemisation and mastery of the rules and relationships involved in equivalence (Ainsworth, 2011).

Another point to note from the video analysis is that when I compared the February video to the July video for each child, there was a reduction in the confusion between the letters d, b, B, p and the number 9 for all children except Kerry who showed no confusion in either video. James and Bethany still showed some confusion between rod letter names in the later video but this was much reduced. For example,

Teacher (July 2016 video): What does that say? (Pointing to $d + r = □$)
James: Green plus red equals something. (Showing a confusion between dark green = d and light green = g)

Another causal code from the video analysis was how it seemed to be the physical moving of the rods that was developing the awareness of the equivalence and overcoming difficulties with the equals sign. I would suggest that by selecting and moving the rods the children seemed willing to take a risk and try a rod in order to
solve an equation, knowing that they could swap the rod for another if it did not work. For example,

Teacher (July 2016 video): What about this one? (Pointing to o – g = □)
Luke: (Gets orange and light green out and lays them side by side. Tries a blue rod and puts it back straight away. Tries a tan rod and jiggles the orange and green around until he is convinced that the answer is not tan. Puts the tan back and swaps it for a black).

This trialling of rods seemed to help them to become confident in knowing when they had the right answer. The rods seemed to be isomorphic and became the model to simplify the process of the algebraic equation and visualise and understand the abstract concept, as suggested in the Literature Review (Goutard, 1963; Post et al, 1977; Kim et al, 2014). It was noticeable how Kerry (most able) was able to solve problems without even touching the rods showing mastery of the structures behind the rods (Goutard, 1964), but would then use the rods to prove her answer. In the later video she said that she visualised the rods when solving problems, particularly involving the three times table, which she was then beginning to master. It was also noticeable how James (least able) was much quicker and confident in selecting the correct rods on the first go in the July video, rather than in the February video. When he did select the wrong rod, he was able to spot it himself most of the time and self-correct, showing him moving from the empirical mode of thought to systemisation by using what he had learnt and experienced about grouping and organising the rods (Goutard, 1964; Ainsworth, 2011).

Referring back to my deductive framework on the different meanings of the equals sign, I would suggest from the video observations that the children were used to seeing the equals sign in different places in an equation. There was also reference to it being called ‘equals’ and ‘equivalent to’ by Kerry and Luke, with James and Bethany only referring to it as ‘equals’. 
Teacher: Can you tell me what this symbol means? (Pointing to = symbol written on the board)
Luke (July 2016 video): Equals … it means it is equivalent to something.

The operational view and sameness-relational view of the equals sign seemed to be well developed as evidenced by the way they talked through the equations (Jones et al, 2013). For example:

Teacher: What does it say to you? (Pointing to = symbol written on board)
Kerry (February 2016 video): Equals.
Teacher: Okay, do you know any other way of saying it?
Kerry: Other ways of saying it … if it’s not there (pointing at the end of an imaginary equation) it’s over there (pointing to where the beginning of the equation would be).
Teacher: Show me what you mean.
Kerry: (Writes 10 = 5 + 5)
Teacher: Okay, what does that mean then?
Kerry: It means ten equals five and five.
Teacher: So you can have equals at the start of it?
Kerry: Yes.

Kerry went on to show she had some understanding of the substitutive-relational view of the equals sign:

Teacher: Are you allowed to have two equals in the same equation?
Kerry (February 2016 video): (Nods)
Teacher: Could you have more than that?
Kerry: Yes you can have more than that.
Teacher: Can you write me one with more than that then?
Kerry: (Writes 5 + 0 and rubs it out. Then writes 4 + 3 and rubs out the 3. Then writes 4 + 1 = 3 + 2 = 2 + 3 = 2 + 2 + 1 = 5) Four add one equals three and two equals two and three equals two add two add one equals five.

This showed Kerry was at least at Chambers’ (1964) fifth step of understanding equality where two rods can equal three rods, but also showed how she was linking elements of her experience together to develop mastery through her “free writing” by writing an equation with four equals signs (Goutard, 1964; Ainsworth, 2011).

Question 4 of the videoed observation was to determine if the substitutive-relational view of the equals sign was understood by seeing if they could make \( y = p + w = r + g \) and could replace the pink and white rod with the red and green rod to equal yellow (Cheng, 2015; Jones et al, 2012). Only Kerry (most able) was able to make the rods to show this equation, again demonstrating that she was the only one who had developed the substitutive-relational meaning of the equals sign, where parts of the equation can be substituted for something else. The other children either put all five rods in a line or assumed the answer was orange or tried to make yellow in different ways. It would seem that these three children were still developing the surface structures of the equals sign, whereas Kerry was developing the deeper structures (Skemp, 1982; Serfati, 2005). Therefore the substitutive-relational view will need further work with these children.

### 4.1.2 Research diary

From the research diary, I could see that most children developed their understanding of the transformation of rods (by this I mean swapping rods around) to make another equivalent equation early on in the research with all children achieving this by the end of the project. This suggests that the surface structures and rules of equivalence were in place for most children early on (Skemp, 1982; Serfati, 2005). Two months into the project, Kerry, Luke and Bethany were able to use equivalence when working with
fractions, showing the deeper structures involved in translating the relationships and meaning from numbers to fractions.

The research diary also showed evidence of James’ confusion between the rod letter names b, d, B, p and the number 9, particularly at the beginning of the research. James was able to record equivalent patterns for yellow, for example, but was still not using the equals sign in his equations. From the research diary I could see that he only began to start to use the equals sign four months into the research project. Six months into the project he was then beginning to use the equals sign in different places, for example at the beginning of the equation. This showed that he was beginning to be able to understand the sameness-relational meaning of the equals sign and start to understand some of the surface structures and rules for using the symbol (Jones et al, 2013; Skemp, 1982; Serfati, 2005). However, he was also using the equals sign in place of an addition sign in some of his equations six months into the research project, showing that he still had some difficulties with the equals sign at that time.

4.1.3 Children’s work

James’ confusion with letter names for the rods and his delayed use of the equals sign could be seen throughout his work, showing triangulation and validating the data. For example, he recorded in his book two months into the project: ‘y p w’ and then told me, “yellow and pink and white,” when he meant that yellow is equivalent to pink add white. A month later, he was learning multiplication by making the same colour trains and writes ‘8 = r’ when he meant that eight is equivalent to four reds, showing that he was beginning to use the equals sign even if the whole equation did not yet make sense. Six months into the project and I could see in his book how he was using the equals sign at the beginning of the equation when he wrote: ‘12 = 3 + 4’ which he explained as “three times four” showing confusion between the addition and multiplication signs.
From looking at the other children’s work, I could see how their understanding of equivalence was developing much earlier than James’. For example from two months into the project, Bethany was writing ‘5 – r = r + w’ and experimenting with fractions and brackets in her equations. In the same time period, Luke was recording ‘2w + 3w = ½ x 0’ and Kerry was recording ‘8 = 2 x 2 + 4’ showing the sameness-relational view of the equals sign where both sides of the equation balance and using the equals sign at the beginning of the equation (Jones et al, 2012).

4.2 Research Question 2

How can “free writing” using letter codes to name Cuisenaire rods, promote an awareness of equivalence?

4.2.1 Video observations

Question 6 of the video observations was designed to enable the children to show and explain some of their own “free writing” about equivalence using the rods. Interestingly all of the children chose not to answer this question when videoed in February but all were able to share some “free writing” in the later videos in July. James (least able) told me in July that, “five lots of two is ten,” and wrote ‘r + r + r + r + r = 10’ and told me that half of orange was yellow, half of tan was pink etc. This showed me that he was at stage 8 of the 9 stages of understanding equivalence (Chambers, 1964). Bethany could use fractions and multiplication in her equivalences in July and wrote equations such as ‘1/2 x 4 = 2’ and ¼ x 8 = 2’ proving them with rods. Luke could do division “nine divided by three is three” and made product crosses, for example a cross of a yellow rod on top of a red rod meant, “five lots of two is ten.” He could also work using fractions and wrote ‘4/4 x 8 = 4r’ proving it with a tan rod and four red rods. Kerry also showed me product crosses and told me that “five quarters of eight is ten” and “four fifths of five equals four”. This “free writing” showed understanding of the surface structures and some of the deeper structures of equivalence (Skemp, 1982; Serfati, 2005).
What the children wrote down when they were “free writing” on the video showed how they were trying to communicate their self-discovery to others to replicate (Cheng, 2015). It can be seen from the examples above, that James’ “free writing” was still at the empirical stage whereas Bethany was at the systemisation stage as she was beginning to link together her experiences of fractions and was reducing expressions with fractions as operators to their simplest form and measuring with a white. Luke and Kerry were working at the mastery stage as they linked their understanding of the elements together in a more complex way (Goutard, 1964, Ainsworth, 2011). This “free writing” on the July 2016 videos showed sameness-relational and substitutive-relational views of the equals sign; particularly where the children were using fractions and substituting them for other fractions (Jones et al, 2013; Jones et al, 2012).

The “free writing” on the video observations was accompanied by explanation and another causal code became apparent during analysis of the children using verbalisation in order to gain understanding. All quotes are verbatim and not corrected into Standard English. For example,

Luke (July 2016 video): Something is the difference between four and white. (Looks at the equation □ = p – w. Gets out a pink and a white and lays them side by side and lays a light green in the gap. Writes 3 = 4 – 1)
Teacher: What does that say then?
Luke: Three equals four the difference between … three equals the difference between four and one.

Kerry (July 2016 video): You can halve dark green to get light green. You can’t halve all the rods like light green.
Here the children were verbalising their ‘know-how’ (Gattegno, 1971) and the surface structures (Skemp, 1982, Serfati, 2005) and used talk to share their knowledge and develop and improve their understanding and belief (Cheng, 2015).

4.2.2 Research diary

From my research diary I could see that Kerry was using brackets, fractions and all four operations in her “free writing” within two months of starting to use the rods. She was also using the equals sign in different places within her equations. Again I could see the deeper structures of mathematical language (Skemp, 1982; Serfati, 2005) being used and the mastery stage of understanding coming out through her “free writing” (Goutard, 1964). I also noted in my diary that the more able children sometimes chose the easy option in their “free writing” and may need to challenge themselves to write more complex equations, which would involve taking risks and experimenting with a variety of representations in order to master them (Baroody et al, 1983; Falkner et al, 1999).

I could also see in my research diary progression in James’ “free writing” and where he began to use the equals sign correctly and where Luke began to use fractions in his equivalent equations. This validated the findings from the July video observations and showed the development of the sameness-relational meaning of the equals sign for all children and the substitutive-relational meaning for some of the children (Jones et al, 2013; Jones et al, 2012).

From my research diary I could also see where other children had introduced new notation to the group. For example, another child had introduced using three rods in their “free writing” within the first two weeks of the project by writing ‘B = w + d + r’, and this new learning was shared with the rest of the class. Also at this time another child wrote ‘4y = 2o’ which meant “four yellow equals two orange,” and introduced the rest of the group to how to write multiples. From my diary I could see that after six months Bethany introduced the need for two equals signs in an equation to the rest of the group, when she wrote ‘2 + o = 2d = 12’ which meant “two plus
orange equals two dark greens which equals twelve”. I could also see that it was after two months that Kerry introduced the need for brackets in more complex equations to help us to understand them. Referring back to the deductive framework, “free writing” can give us the fertile situations which promote reasoning (Mueller et al, 2015), and it is the skill of the teacher to perceive their children’s awareness’, ask the right question and then step back and allow the children to discover the need for symbols for themselves, helping them to clearly express themselves through their “free writing” (Gattegno, 1963, 1971 and 1974; Goutard, 1964). Through talking with children as they “free write”, a teacher can analyse errors and their thinking, and correct misconceptions (Prediger, 2010). It is therefore important for teachers to have diagnostic competence and a deep pedagogical awareness of how children learn mathematics and how to make concepts accessible to students (Shuman, 1986).

4.2.3 Children’s work

From looking at the children’s work, I could see a progression of their understanding through their “free writing” which triangulated and validated the data. I could see where I had shown economy of learning (Gattegno, 1971) as I guided Bethany to write using brackets and fractions for the first time as she recorded ‘y – 2(1/2 x r) = g’ which she explained meant, “the difference between yellow and two lots of half of red equals green.” Seeing expressions written in a variety of ways helped children to understand them, make sense of them and master their notation and Bethany could then be seen using similar expressions further on in her work (Baroody et al, 1983; Falkner et al, 1999).

Again, in their work, I could see the importance of children verbalising to a teacher in order to gain understanding and self correct mistakes. I had noted in Luke’s book where he spotted the missing two in front of the brackets himself as he explained his expression to me, and was able to correct it to read ‘1/2 x o – 2(1/2 x p) = ½ x r’ which he explained as, “the difference between half of orange and two times half of pink is equivalent to half of red.”
Through looking at evidence in their books I could see where Luke and Kerry started to introduce two equals’ signs into their equations and where Kerry introduced the need for a remainder to her division equations. As a teacher my role is to promote the subordination of teaching to learning; therefore I need to have an awareness of what they are “free writing” in order to take a step back and then ask the right questions in order for new knowledge to be discovered, and to promote an economy of learning rather than repeated learning experiences (Gattegno, 1971).

From looking at their books I could see how the children had developed through Chamber’s (1964) nine steps to an understanding of equivalence. James had progressed as far as step 8 where he knows that a rod can be made up of a train of the same colour rods e.g. \(10 = 5 \times 2\), but the other three children had all shown evidence in their books of the final step where trains of rods of one colour can equal trains of rods of another colour e.g. \(5 \times 2 = 2 \times 5\).

From looking at their books I could see that the children can access far greater mathematics than at times we think they are capable of (Goutard, 1964). I could also see how a well timed conversation with the teacher during learning can move their understanding on so that they can include the deep structures of mathematical language as they try to reason and verbalise their “free writing” (Goutard, 1964). For example, Kerry wrote in her book ‘\((2/5 \times 5) + (4/4) = 9\)’ and when I asked her to explain her reasoning to me, she could quickly see her error and corrected her equation to read ‘\((2/5 \times 5) + (4/4 \times 4) = 9 - 3\)’. She then explained to me that this meant, “two fifths of five plus four quarters of four are equivalent to the difference between nine and three.” Here I could see how she had linked together understanding of different concepts into her expression, showing the mastery stage of a much deeper understanding (Goutard, 1964, Ainsworth, 2011).

4.3 Research Question 3
how can working with different media, each corresponding to different representations of equivalence, help teachers and learners move fluently between representations of equivalence?

4.3.1 Video observations

From analysing the videoed observations, I found that only Luke (average ability) and Kerry (most able) used other media and representations of equivalence to help them to answer my questions. In the July 2016 video, Luke was using his fingers to count in twos to find out what five quarters of eight was when he had taken out five red rods. In the February 2016 video of Kerry, she used her fingers to prove the answer to \( d + r = \Box \), for example:

Kerry (February 2016): (Gets out a dark green and a red rod and places them end to end). It means you are going to add a red onto dark green. Then it makes tan (lays a tan rod side by side with the dark green and red). You can make it with the rods or work it out on your fingers. Because six (puts up six fingers) add two (puts up two more fingers) makes eight.

She also used her fingers in this video to prove the answer to \( \Box = p + w \) by counting on her five fingers. She also explained the transformation of rods:

Kerry (February 2016): Pink add one equals five and you can swap them round and white add pink equals yellow.

She explained later in this video that you can, “make it out of rods or work it out on your fingers,” showing that she was confident with both representations.

In the July 2016 videos Kerry and Luke both showed they were confident to use product crosses as a different representation of multiplication using the rods, for example:
Luke (July 2016): (Lays a yellow rod on top of a red rod in a product cross) Five lots of two.

Kerry (July 2016): (Makes a product cross of light green on top of pink) Three times four is twelve. (She swaps the light green and pink over so the pink is now on top). Four times three is twelve.

These two children were beginning to move fluently between trains of one colour and product crosses as different representations of multiplication.

In the July 2016 video Kerry also demonstrated fluency with other representations such as gesturing with her hand to cut a tan rod in half (probably linked to practical work we did previously on fractions by cutting fruit and dough), and using arrays for division:

Kerry (July 2016): Ten divided by five. (Gets out two yellows and lays them side by side with the orange. Gets out five whites and lays them in a row spaced out and not touching. She lays a red under each white to form an array with the whites probably representing five people to share ten between, giving them a red each. She counts aloud as she lays each red) 2… 4… 6… 8… 10.

All children by the end of the project showed that they could move between using the rods, and writing equivalences using letter names or numbers, some of whom could do this without using the rods at all. I would like to further develop these representations to include an interactive set of rods that could be manipulated on the screen as a stepping stone from actually manipulating the real rods to being able to write equations without touching the rods at all but only visualising them; moving from the concrete to the pictorial as an aid to moving to the abstract.
Referring back to the Literature Review, I could see from these examples mastery of the steps to understanding equivalence (Chambers, 1964) and mastery of the structures behind the rods (Goutard, 1964). I could see a wider understanding of equivalence where it is possible to replace one item by another (Gattegno, 1974) and both the sameness-relational view of the equals sign in Luke’s use of the product cross, and substitutive-relational views of the equals sign in Kerry’s use of the product crosses to make twelve (Jones et al, 2013; Jones et al, 2012). This ‘know-how’ becomes really powerful when children such as these, can move fluently and confidently between representations of equivalence.

4.3.2 Research diary

Within the first few weeks of this research project I noted down in my research diary that James had said that, “a ruler equals keywords add a tape” (Appendix C), showing that the children were translating their knowledge of the equivalent length of rods to the equivalent length of real life objects, demonstrating a sameness-relational view of equivalence (Jones et al, 2013).

I also noted down in my diary early on where children were struggling with concepts such as equivalent difference, moving fluently between letter names and number names for the rods, remembering the number names for each rod and where they were struggling to move between multiplication and the equivalent array. I promoted the subordination of teaching to learning (Gattegno, 2010) through an economy of learning at the time when they struggled, by moving on through Gattegno’s (1963) Textbook 1 to new learning experiences. I knew that the children would meet these concepts again and would build on their understanding through using different media and representations of equivalence as we worked through Textbook 1.
Six months into the research diary I could see where all of the children had moved their understanding on and were able to solve missing number equations involving different operations, fractions, brackets and the equals sign. By teaching the four operations and fractions as an operator all together, it seemed to have allowed the children to see the relationship between them and develop and link their understanding (Gattegno, 1971 and 1987). I could also see where children were checking their rod equations on their fingers and were able to use the rods to solve real life problems. Also at this time, I could see where most of the children could test a number to see whether it was odd or even, prime or composite by using their rods to check the factors of the number. By composite I mean the opposite of a prime number, having more than two unique divisors. Some children including Kerry and Bethany were also using the transformation rule of swapping numbers around in multiplication equations without having to check the answer was correct, showing mastery of the commutative structures behind the relationships between the rods (Goutard, 1964; Ainsworth, 2011).

Towards the end of the project, I could see from my notes in my diary how easily most of the children transferred their knowledge of equivalence to comparing equivalent length, capacity, time and weight. Most of the children could also use their rods to make equivalent money amounts using the orange rod to represent 10p, the yellow to represent 5p, the red to represent 2p and the white to represent 1p. They could also all make two digit numbers by only using orange and white rods.

Using Gattegno’s (1963) Textbook 1 helped me to make them conscious of relationships and structures between the rods (Goutard, 1964), and this appeared to help them to begin to move fluently between different media and representations of equivalence as detailed in the examples above. The exercises in the textbook helped them to gain mental structures and link together mathematical concepts (Gattegno, 1963; Coles, 2014). They could then apply this to real life problems and other areas of mathematics such as measures.
4.3.3 Children’s work

From James’ book, I could see where he was beginning to use reasoning to apply his knowledge of the relationship between the rods to prime numbers, saying that six was not a prime number because, “it can be made out of reds.” He was beginning to move between trains of one colour showing the factors of a number and an understanding of what a composite number is. All of the other children could also transfer this knowledge of factors and equivalence to decide whether a number was odd or even, prime or composite.

From Bethany’s book, I could see two months into the project how she could easily transfer her knowledge of equivalent equations to include fractions in them, recording expressions such as ‘5(1/7 x b) = y’ meaning, “five times one seventh of black is equal to a yellow.” Six months into the project she was easily swapping numbers in multiplication equivalences without having to check the answer, as she seemed to be transferring her early work with the rods and equivalent multiples or trains of one colour to a new representation of multiplication.

From Luke’s book, I could see early use of fractions and the substitutive-relational view of the equals sign as he recorded ’13 = 8 + 5 = 7 + 6’ and ‘(1/2 x 14) + (1/2 x 14) = 14’. He could also move fluently between knowledge of equivalence, factors and multiples by reasoning why 29 is a prime number and 6 is not, by showing me that 6 can be made out of red rods or light green rods.

Kerry recorded in her book many examples of making the equivalent of thirteen, moving fluently between fractions and all four operations as she wrote ‘(1/2 x 8) + (1/2 x 8) + 5 = 13’ and ‘(1 x 5) + (4 x 2) = 13’. She also transferred her knowledge of equivalent length to real life objects as she recorded ‘pencil = glue stick’, ‘pencil > rubber’.
Referring back to my deductive framework, I can see how the use of rods to teach equivalence can allow children to transfer their deep understanding of one representation to other representations such as arrays, finger counting, product crosses, written algebraic and arithmetic equations and real life problems. I would like to develop this further in subsequent years to include more representations and a greater fluency between them, as required by England’s 2014 National Curriculum. I can also see how developing the substitutive-relational view of the equals sign in subsequent years could help children to substitute one representation for another more easily. By making children aware of other representations and media for equivalence, they could gain more ‘know-how’ and link together these different representations to gain mastery of mathematical concepts, as also required by England’s 2014 National Curriculum.
Chapter 5: Conclusion

The overall aim of my research was to reflect on the use of Cuisenaire rods to teach equivalence. The objectives were to explore the children’s understanding of equivalence and their use and understanding of the equals sign, and to reflect on Gattegno’s model of mathematics teaching in relation to equivalence. I believe I have achieved those objectives. I also intended to discuss whether this model of teaching and learning mathematics would have a positive impact on conceptual understanding and I believe I have shown that it has. This thesis contributes to an important area which I believe needs to be researched further.

5.1 Methodological evaluation

This research design was an effective way to study this area as through action research I was able to evaluate the impact of this style of teaching using Gattegno’s model by collecting evidence of the impact of this approach on the children’s understanding of equivalence. I believe I was right to weight my evidence in favour of the video observations as they were rich sources of data that could be analysed, and which bought about the thematic and causal coding I used. Triangulating and verifying the observations using children’s work and my research diary proved important, and analysing all data against the deductive framework set up in my literature review was an effective analytical tool. I believe I am right to be cautious about my findings due to the very small sample size which was selective in terms of providing a range of abilities. The findings may also not be repeatable and thus not reliable. Although for this particular study the research design was effective, I would modify it for further use by increasing the sample size. In subsequent research in this field, I do not believe that a research diary and children’s work would both be needed as they provided in essence the same evidence.
5.2 Research Questions

5.2.1 Research Question 1  
How can Cuisenaire rods be used to promote an awareness of equivalence and help overcome difficulties with the use of the equals sign?

Evidence from this research shows that the rods can promote an awareness of equivalence through the process of selecting and moving the rods to attempt to solve missing number equations, allowing children the chance to risk getting it wrong and self correcting with a different rod. The rods seem to give the children a model for visualising and understanding abstract concepts. The rods seem to help overcome difficulties with equivalence through promoting the sameness-relational view of the equals sign for all children in the research, and the substitutive-relational view of the equals sign for some of the more able children. The deeper structures and rules of equivalence can be seen, through looking at the relationship between the rods.

5.2.2 Research Question 2  
How can “free writing” using letter codes to name Cuisenaire rods, promote an awareness of equivalence?

By “free writing” the children have no constraints on what or how to record, and are trying to communicate what they have discovered for themselves about equivalence, so that others can replicate it. This “free writing” again can show both sameness-relational and substitutive-relational views of the equals sign. However it seems to be when the children are verbalising their “free writing” in order to explain their knowledge and gain understanding, that the role of the teacher becomes crucial in order to expose them to more complex ways of writing equations through well timed
intervention and well placed questions to move their learning on. Through this verbalisation as they “free write”, children can discover the need for notation for themselves and teachers can diagnose errors and correct misconceptions. It is through this “free writing” that we can see the complex equations that children are capable of, and mastery of a much deeper understanding.

5.2.3 Research Question 3

How can working with different media, each corresponding to different representations of equivalence, help teachers and learners move fluently between representations of equivalence?

Children’s knowledge of the mastery of deeper structures and relationships when using the rods can become really powerful when they can move fluently between different representations of equivalence. Many of the children could move from the concrete of moving the actual rods, to the abstract use of the letter names when writing equivalences, and I would like to develop this further in the future to include using interactive pictorial representations too. By teaching all four operations and fractions as an operator at the same time, it seemed to allow the children to make more links between these elements and therefore gain mastery of mathematical concepts.

For the children who had developed a substitutive-relational view of the equals sign, it seemed easier for them to move between different representations of multiplication for example, using same colour trains, product crosses and arrays to solve problems. These children also seemed to be able to apply their knowledge of equivalence to equivalent measures and to help them solve real life problems.

5.3 Implications for further research

Specific implications for further research outlined in this thesis would be:
• Research conducted in the other three schools I am working closely with who are also adopting the Early Algebra approach through their involvement with Sociality Mathematics.
• Research into the perceived successes and benefits in participating in this Gattegno model of teaching for those teachers and schools which have participated for some time.
• Research that would track this model over a number of years.
• Research into how Cuisenaire rods could develop other concepts such as difference or fractions.

Such research should be of interest to those working in the field of algebra in the primary classroom beyond the scope of this research, and those working on the implications of England’s 2014 National Curriculum.
Appendix A: Questions used in the video observations

Questions for Final Video July 2016

1. What does this symbol mean? Has it got more than one meaning? Can you give me an example of how to use it in different ways?

= 

2. What does this mean? Can you show me with the rods?

\[ d + r = \]

3. What does this mean? Can you show me with the rods?

\[ o - g = \]

4. What does this mean? Can you show me with the rods?

\[ = p - w \]

5. What does this mean? Can you show me with the rods?

\[ y = p + w = r + g \]

6. Show me some of your own free writing using the = symbol and explain it to me using the rods:
Appendix B: Example of video observations transcription

Video Analysis: Y1 (J) July 2016

Teacher: What can you do with the rods?
Pupil: I can ... I can halve them.
Teacher: Can you show me that?
Pupil: (Takes 2 dark green rods out of the box and lays them on the whiteboard. Moves them apart form each other. Then gets another 2 dark greens out and joins them on to the 2 already split up on his board).
Teacher: What’s half of dark green?
Pupil: (Gets out 2 light greens)
Teacher: Show me the rod which is half of tan.
Pupil: (Gets out a tan and 2 pinks and throws them on the board)
Teacher: Show me the rod which is half of orange.
Pupil: (Throws a yellow rod onto the board)
Teacher: Do you know any other ones?
Pupil: (Throws a pink on the board)
Teacher: What’s half of pink?
Pupil: (Pauses before throwing a red on the board)
Teacher: And what’s half of red?
Pupil: (Throws another red on the board and then a light green)
Teacher: What’s half of red?
Pupil: (Shows me a pink)
Teacher: Put those rods away. (Pausing while he clears his board) Can you tell me what this symbol means? (Pointing to the = symbol on a whiteboard)
Pupil: Equals.
Teacher: Anything else?
Pupil: (Pauses) No.
Teacher: What does that say? (Pointing to d + r = □)
Pupil: Green plus red equals something.
Teacher: Can you show me that?
Pupil: What? On the board?
Teacher: Do it with your rods first.
Pupil: (Throws a tan on the board)
Teacher: Is that the answer?
Pupil: Yes.
Teacher: How do you know that is the answer?
Pupil: I guessed.
Teacher: Do you want to check then? Leave it out and check with your d and your r.
Pupil: (Lays a red alongside the tan)
Teacher: What’s the d for? D?
Pupil: (Picks up a tan, then puts it back and gets a black. Then puts the black back and puts a dark green in the gap end to end with the red.)
Teacher: Were you right?
Pupil: Yes.
Teacher: Fantastic. What about that one? What does that one say? (Pointing to d − g = □)
Pupil: (Throws an orange on the board) Orange and green equals (Throws a dark green next to the orange and then puts a pink into the gap)
Teacher: What’s the answer then?
Pupil: Pink.
### Appendix C: Example of an extract from my research diary

<table>
<thead>
<tr>
<th>Who</th>
<th>What I learned</th>
<th>Significance of learning</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>They can swap rods</td>
<td>RQ1: Transformation of rods for equivalence</td>
</tr>
<tr>
<td></td>
<td>around to get another equivalent</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Good at knowing what equivalent rod is</td>
<td>RQ1: Can see missing rod</td>
</tr>
<tr>
<td></td>
<td>$B = w + e$</td>
<td>RQ2: Free writing</td>
</tr>
<tr>
<td></td>
<td>Using 3 reds</td>
<td></td>
</tr>
<tr>
<td></td>
<td>None could use word &quot;different&quot;</td>
<td>RQ3: Not moving fluently between representations yet</td>
</tr>
<tr>
<td></td>
<td>when talking about its rod</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Could apply equivalence to real life objects</td>
<td>RQ3: Apply to real life objects</td>
</tr>
<tr>
<td></td>
<td>Ruler = keywords + tape</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Introduced $4y = 20$ to go + can be done in any order</td>
<td>RQ2: Free writing promotes new ways of recording</td>
</tr>
<tr>
<td></td>
<td>$y = 5$</td>
<td>RQ1: Transformation</td>
</tr>
<tr>
<td></td>
<td>Secure of using word &quot;different&quot;</td>
<td>RQ3: More able moving fluently between representations</td>
</tr>
<tr>
<td></td>
<td>Really good at missing no sums</td>
<td>RQ1: Missing no sums</td>
</tr>
<tr>
<td></td>
<td>Using &quot;difference&quot; now</td>
<td>RQ3: More able are moving fluently between representations</td>
</tr>
<tr>
<td></td>
<td>Only one writing use ( ) and missing rods</td>
<td>RQ2: Free writing</td>
</tr>
<tr>
<td></td>
<td>Assigning numbers to rods</td>
<td>RQ2: Free writing</td>
</tr>
<tr>
<td></td>
<td>Only one writing with fractions</td>
<td>RQ3: Need to use different representations</td>
</tr>
</tbody>
</table>
Appendix D: Example of children’s work in books

Put in the right place.
Write lots of different equations for that colour rod.
Remember the number names if w=1

Deepening: Set a question for a friend – true or false, connections with my

\[
\begin{align*}
\gamma - \delta &= \gamma + \delta & \checkmark \\
\gamma + \delta &= \delta & \checkmark \\
\gamma - \delta &= \gamma + \delta & \checkmark \\
\gamma - \gamma &= \gamma & \checkmark \\
9 + 9 + 9 &= 9 + 9 + 9 & \checkmark \\
\gamma - \gamma &= \gamma + \delta & \checkmark \\
\delta - \delta &= \delta & \checkmark \\
(\gamma - \frac{1}{2} \delta) &= \frac{1}{2} \delta & \checkmark \quad \text{WOW!}
\end{align*}
\]

\[
\begin{align*}
(\frac{1}{2} \delta) + \gamma &= \frac{1}{2} \delta & \checkmark \quad \text{WOW!}
\end{align*}
\]
Appendix H: Findings for participants

July 2016

Dear Parents and Year 1 children,

Last December 2015, I wrote to explain the research project I was undertaking for my dissertation to complete my Masters in Mathematics Education and to ask your permission to include video evidence and work in books from your children. I wanted to find out if using Cuisenaire rods in every mathematics lesson would have a positive impact on children’s understanding of equivalence and help overcome difficulties with the meaning of the equals sign.

My results showed that the rods can promote an awareness of equivalence through the process of selecting and moving the rods. The rods give the children a visual model for understanding abstract concepts such as the rules of equivalence. When the children were given opportunities to “free write” with no constraints on what or how to record, they demonstrated they were capable of a much deeper understanding of what the equals sign means and wrote complex equations of equivalence involving brackets, all four operations and fractions as an operator. Children’s knowledge became really powerful when they could move fluently between manipulating the rods to writing abstract equations using the algebra of letter names for the rods. The children and I shared our algebra work with some parents and governors at a workshop at the end of June, giving them a chance to see first hand what the children had achieved. I hope you all have had a chance to look at some of the work in their books throughout the year to see the remarkable results of what they have been able to achieve.

I would like to thank the children for their fascinating insights and as a staff we will be continuing this work using the rods as they move into Year 2.

Yours sincerely

xxx