3 Reflections

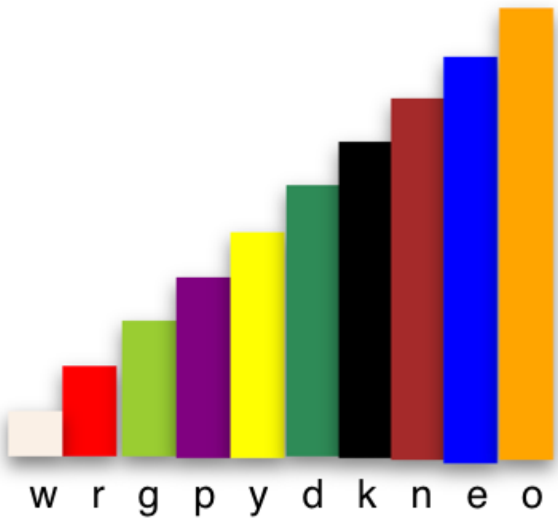
On the Black Rod

Ian Benson

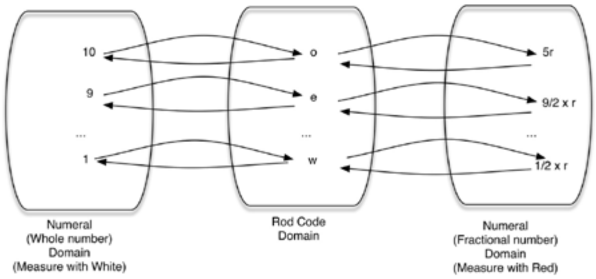
Gattegno and his collaborators embarked on a programme to apply ideas about classes and their connections, from conceptual mathematics, to reformulate school mathematics (*Choquet, 1963*).

Conceptual mathematics is concerned with links between 'sets' that are already structured: a set being a collection of elements, and a structure being a relationship between the elements (*Lawvere & Schanuel, 1997 and Benson, 2016*).

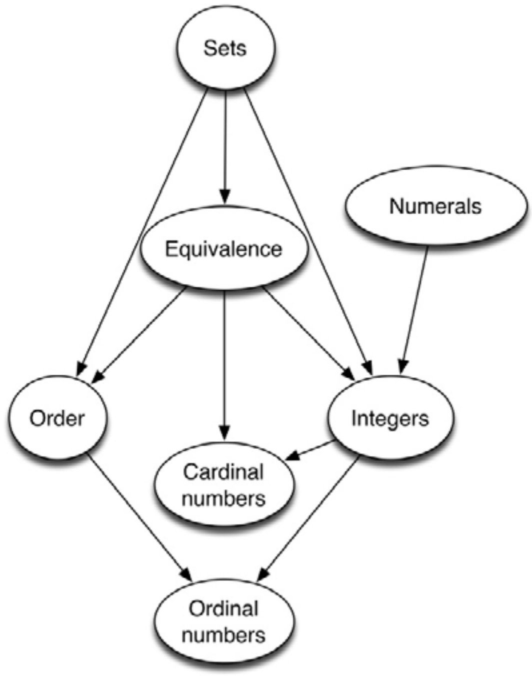
Two such sets are said to be isomorphic if there are mappings from one set to the other such that each element is mapped to a single element of the other while any relationship between elements is transformed into a relationship between the corresponding elements. The rods have a structure that is captured by the 'staircase'.



The rods (and these structural relationships) can be mapped onto the integers. They can also be mapped onto multiples of any other rod (in the example below, multiples of the red rod) (*see Trivett, 1963*).



We can record maps between these sets in a functional relationship diagram (see above) (*Benson, 2015*). When these functions preserve the order relationship in the staircase they are called isomorphisms.



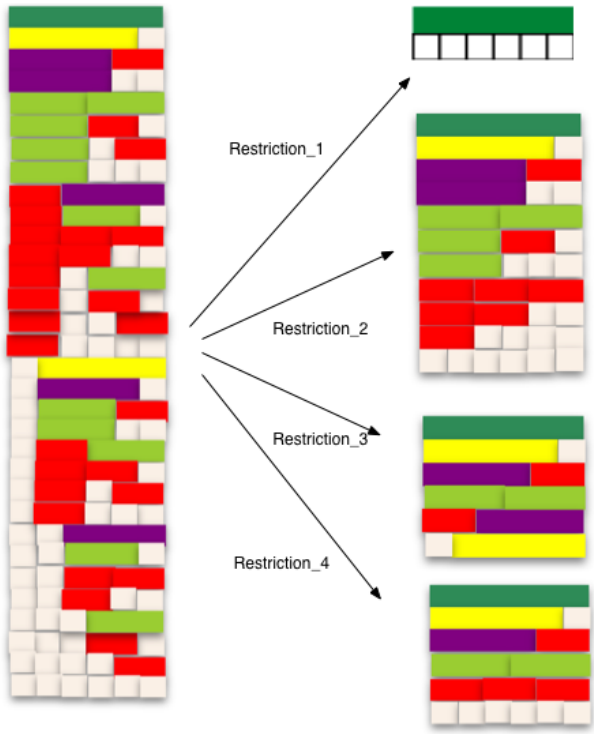
In this writing I will explore an activity to educate awareness about these ideas. Gatttegno wrote of the toddler, "I already know a lot about classes and isomorphism simply from looking at the world around me where nothing is ever seen under the same light, from the same distance or at the same angle. I recognise everything in spite of a multitude of transformations affecting it." (*Gattegno, 1970b, p4*). The mathematics teacher's challenge is to develop a pedagogy that harnesses such mental powers.

Gattegno drew attention to special functionings which mathematicians use: awareness of relationships per se, and awareness of the dynamics of the mind (*Gattegno, 1987b*). This idea of mathematical awareness was central to his project to reform the curriculum based on the study of permutations and combinations of colour coded Cuisenaire rods.

In what follows, I aim to study and extend a curriculum chart, a mechanism proposed by Gattegno to educate teachers' awareness of the temporal hierarchy of concepts, that comprise this new curriculum. By temporal hierarchy Gattegno meant the logical order imposed upon these awarenesses by the dependence of one upon the other. In his curriculum chart dependencies are represented as arrows between nodes labelled by concepts (*Gattegno, 1974*).

The diagram above shows a fragment of his chart that deals with awarenesses that can follow from a study of trains of all white Cuisenaire rods (in code w,w,w…w) equivalent in length to a rod of length L. He calls this Restriction_1 of the complete pattern made by all permutations and combinations of rods equivalent in length to a rod. The diagram below on the left is this "complete pattern" for the Dark Green rod. On the right are the various segments of the pattern that link to Gattegno's 'Restrictions' and from which he showed different and important elements of mathematics could be drawn.

Restriction_1 shows the all White train equivalent to the length of a Dark Green rod (code d). It has 6 cars, so when the time comes Dark Green can be called "6 measured with a White."



First though the learner will explore the complete pattern for as many rods as they can. They discover through free play and free play guided by questions in his books that the rods can be characterised in a number of ways (*Gattegno, 1960*):

- rods of the same colour are the same length (and vice versa)
- rods of different lengths are different colours (and vice versa)
- the length of every rod can be made by a train of multiple White rods.

In some circumstances we can combine the lengths of two rods to make a length equivalent to the length of a third rod. We therefore restrict the complete pattern (all the ways we can make a rod of a given length) to just those combinations using two rods; Gattegno systematised different ways that the complete pattern can be restricted, to force different awarenesses. The restriction of only using two rods is labelled Restriction_3.

Gattegno calls the situation this restriction generates, in which two objects can be combined to form a third object of the same kind, an algebraic relationship. Not all relationships are algebraic. For example the “greater than” relationship between rods of different colours is simply True or False and not linked to another rod (*e.g. Gattegno, 1963*). Activities that comprise “algebra before arithmetic” take up the first few chapters on his Year 1 textbook (*Gattegno, 1970a*).

As mentioned above, Gattegno calls the set of all permutations and combinations of rods equivalent to a rod (or train) a complete pattern. Given a set of distinct elements like the trains in a such a pattern, we can sometimes define an equivalence relationship between the elements.

Equivalence is a relationship between objects with a common property. We say that two such objects are equivalent just in case that property has the same value for each of them. For example, two red rods are distinct elements of a set of rods, but they are equivalent colours.

More generally we can say two objects are equivalent if there are circumstances in which one of them can be substituted for the other. In this case we often indicate the circumstance by qualifying the type of equivalence.

Gattegno identifies four types of equivalence in his textbook for the rod world for Year 1: equivalence of colour, length, form/name (that is, algebraic expressions naming trains of equivalent length), and expression (that is, arithmetic expressions with the same value) (*Gattegno, 1970a*). Each of these equivalences corresponds to a domain of values: the physical rods (with the action of sorting by colour), mental imagery (with virtual actions of juxtaposition to form new lengths), algebraic expressions (from which we construct equations), and arithmetic expressions (from which we also construct equations).

Gattegno writes:

> “It is obvious that by giving children a set of objects upon which it is possible to define an operation, we can enable them to discover what propositions follow from that simple fact. Fortunately ... Cuisenaire has produced a set of coloured rods where the operation of placing them end to end is ‘isomorphic’ to addition and a multitude of propositions can be obtained from such simple situations as having two equal lengths and forming any length by placing rods end to end, without any mention of numbers. This can be easily done by children of five, and they have no doubts about the propositions obtained. For the mathematician this is algebra; for the child it is a meaningful game which can be varied indefinitely” (*Gattegno, 1963, p63*).

In their study of Gattegno’s theory of learning, Young and Messum argue that “we can exhaustively identify the awarenesses needed in any domain and redefine teaching as the activity which leads students to cover this ground for themselves without missing any essential steps and without wasting time. Gattegno demonstrated this with thousands of students in hundreds of classrooms around the world” (*2011, p16*).

But the mathematical world is unbounded. It has an infinity of awarenesses. Conceptual mathematics provide the technical machinery to subdivide it into domains, so called sets with structure. Some of these domains will be finite and some will have an infinite number of elements. The skill of subdividing the world in this way, stressing some aspects while ignoring others, is the process of mathematising. I describe activities to force awareness of the concept of set, set with structure, equivalence, integer, cardinality, order, isomorphism and automorphism (G*attegno, 1956, p. 85, 1963, p52*). An automorphism is an isomorphism between a set with structure and itself.

Gattegno wrote:

"Mathematics has a unique role to play in education, because its main function is to prepare everyone who cares about it to acquire its techniques for the job of making of mental models of situations.. and deducing from these models conclusions that were hidden but which are needed in order to forge ahead... It is a power given to the individual so that he can find more in what he meets. This power is the birthright of every one of our children, and schools should make it accessible to all" (*Gattegno, 1958, p51, 1963, p90*).

The Science of Education Working Group studied the structure of the set of four car trains equivalent in length to the Black - I invite readers to try this for themselves before continuing. It is instructive to group trains (as in the task above), in the complete pattern, into sub-sets or classes according to the number of cars in each train, i.e., one car trains, two car trains, etc.

The national curriculum

Traditionally arithmetic is built on cardinality, a concept Gattegno suggested in his curriculum chart was itself dependent on sets, equivalence and integers. By integer, or whole number, Gattegno meant the numeral associated with a complete pattern once its algebraic structure had been thoroughly explored though physical and virtual action and isomorphic algebraic writing (*Gattegno, 1970, pp1-30 and 1974*).

Making cardinality the foundation for arithmetic in the early years and Year 1 curriculum is not a good foundation for mental arithmetic. Students learn the "counting poem" and enumerate the elements in a collection (fingers, objects, illustrations). They associate a unique numeral name with each element. The cardinal number is the name of the last element reached. Addition is modelled by counting on, and subtraction by counting back. A refinement of this method requires learners to memorise the partition of numbers up to 100 into two parts.

All these processes are slow and error prone. Students are faced with unlearning these methods in order to become fluent in mental arithmetic. Gattegno wrote:

"The methods we have inherited in the West succeed in making the intellect sharper and more penetrating, but they fail to provide a succession of springboards by means of which the mind can become ever freer of the restrictions it meets. The old methods are directed towards disciplining the mind; what is needed today is an education that leads to greater awareness of what constitutes the mind and of how to master the dynamics of thought, of the emotions and of creativeness" (*Gattegno, 1958, p51, 1963 p90*).

To Gattegno, the traditional curriculum is built on a defective foundation. The 2014 UK curriculum recognised this. It is one of the first in the world to propose an entitlement in Year 1 to learn all four arithmetic operations and fractions as operators together for small numbers. It is ironic that the curriculum designers, having willed the end in this way, continue to promote counting as a foundation and fail to insist on early algebra - a proven way to cover this ground.

The subordination of teaching to learning

The subordination of teaching to learning in mathematics is about how teachers help students to learn. This involves mathematising situations. The teacher's role is to create energising games and exercises which make learners aware of certain concepts and to provide conventional labels and notations which students cannot invent themselves. This is called the phase of awareness. The student's role is to engage in dialogue with the situation for a period of time to yield as much as possible and to acquire know-how in using various skills. This is called the phase of facility (*Catir & Gattegno, 1973, p3*).

Gattegno suggested through his curriculum chart a way to create mathematising situations through the study of permutations and combinations of Cuisenaire rods. First published as an appendix to ***For the teaching of mathematics*** (*Gattegno, 1963, p185*) it was revised in his ***Common Sense of Teaching Mathematics*** (*Gattegno, 1974, p193*). Piers Messum and I have updated it as a curriculum graph in (*Cane, 2017, p4*).

Gattegno's curriculum chart has four "root" nodes - each a sub-set or restriction of the complete pattern of all trains equivalent in length to a rod. Gattegno defines the restrictions (R) as follows (see diagram above to make sense of these categories, in relation to the Dark Green rod):

R_1 Length measured by train of White cars (carriages).

R_2 Indifference to the order of the rods in each train. Illustrated here with trains ordered (i) left to right by non-increasing length of cars, and then (ii) top to bottom by non-increasing length of cars in the first column, and if necessary in subsequent columns.

R_3 Equivalent trains with two cars

R_4 Equivalent trains including (i) cars of one colour, or (ii) cars of two colours including a single smaller rod. This is a further restriction of R_2.

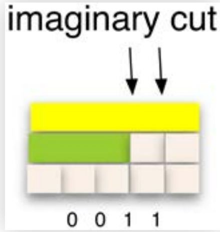
Almost every node in his chart has a path back to at least one of these restrictions. These paths direct the teacher's attention to the pre-requisites for each new idea. In his text-books he illustrates each concept in turn though open questions designed to encourage the learner to investigate for themselves some rod construction or related expression or equation. Becoming aware passes through several phases: unconscious, conscious, named, and categorised.

As a book for teachers ***Common Sense*** uses more developed mathematical language than the vocabulary in his textbooks for students. Teachers are expected to understand mathematical ideas such as set, function, operator, permutation, algebra of sets, correspondence, numbers in different bases, field, ring and polynomial. Starting with actions with the rods to construct patterns of trains of equivalent length, Gattegno forces the learner to become aware of (i) physical manipulation and sorting (ii) virtual action (in the mind) to visualise these patterns (iii) naming constructions with colour names, letter codes and expressions involving all four arithmetic operations and fractions as operators and (iv) arithmetic expressions and equations.

Numerals appear first as multipliers in algebraic writing. He writes, "we can produce six glasses of beer from a cask. Six is here a numeral when the operation is over, but it is an operator while the drawing is taking place. .. Hence we must add to the various meanings .. that of number as an operator, although it is still not yet a mathematical entity." (*Gattegno, 1971, p4*)

Rods and trains are initially given symbolic names, with numerals first appearing as multipliers (adjectives or operators). We use different forms for a name to highlight distinct awarenesses.

- Goutard names a train from the elements that make it up. For example, o b o (*Goutard, 1964, p41*)
- Gattegno names a train by the action of placing rods end to end, o + o and by using the numeral as an operator, 2o (read as "two lots of Orange") (*Gattegno, 1970, p22, 25*).
- In my paper train notation I concatenate the letter codes to direct attention to the train as a single entity obo (*Benson, 2011, p17*)
- Using the Haskell language we can name a train with a value of type String, e.g. "bbb".
- We can give each train a binary label (a numeral made up of digits 1 and 0). This is created by considering the pattern of cuts in a rod of length L which needs to be made to create the train length L. The figure shows the process of "cutting" the Yellow rod using the all White train (1111) as a grid. Reading the White rods from left to right we append a 1 to the right of the label if the right hand edge of the rod aligns with a cut, otherwise we append a 0.



- We can name each train with an integer in binary, or decimal, by reading the binary label as a number. The naming function is known as an indicator function. It induces an order on trains of the same length with the same number of cars.

A study of four car trains

This activity develops a task suggested by Arthur Powell. He uses it to introduce teachers to Gattegno's approach.

The questions posed are:

1. Make as many trains as you can with four cars that are equivalent in length to a Black rod.
2. Compare your construction with a neighbour ... have you got them all?
3. How do you know you have made all possible trains?
4. Study the various naming conventions for a train. Which convention is more efficient in helping you to describe your thinking?
5. The indicator function induces an order on the four car trains. Complementing the binary label (replacing 0 by 1 and vice versa) maps the set of four car trains to itself. Can you show that this mapping is an automorphism with respect to the induced order?

This exercise is sufficiently challenging to ensure that students need to work systematically if they are to be confident in making all the possible trains. At the Science of Education working group, after 10 minutes, none of us had managed to get beyond step 3.

Isomorphism is key to understanding Gattegno's use of algebraic writing as a gateway to number.

In Numbers in Colour the number name for a rod is not attributed to it until the complete pattern of trains equivalent in length to the rod have been studied through action, speaking and algebraic writing. The three domains, ideographic (imaginary patterns and their restrictions), algebraic and arithmetic are considered as formal languages. Constructions in one language can be mapped into constructions in another through mappings that preserve relationships (morphisms).

The constructions in the rod world are necessarily limited by the number of available rods. Mental imagery, written algebra and arithmetic have no such limitations. Thus action with the rods acts as a catalyst to subsequently shift attention from a finite world to mappings between domains each of which enjoys an infinite dynamic.

For teachers and students the Black rod activity poses an exercise in the role of labelling and representation in educating awareness about permutations and combinations. It explores the distinction between action with the rods, virtual action (experienced as functions within and between domains), and awareness of that virtual action in Step 4 by naming or specifying the function to transcode between a domain (the rod world) and a codomain (computer code).

Reflections from a meeting

Alistair Bissell

Since the meeting on 7/12/13, the meanings of terms, such as 'ignore', 'stress', 'awareness', 'noticed', 'focusing attention', 'pending' and 'marked', continue to shift for me as I reflect on my experiences on the day. It became apparent to me on the day that careful and accurate use of these terms is important to the group. I suspect that in the process of this very writing, the meanings of these terms will continue to shift for me, yet it is these terms with which I am to describe my experiences and thoughts. One of the most powerful experiences for me on the day came from being challenged on my use of the term 'ignore', and so I invite further challenges to my use of these terms in the writing that follows.

Activity: What does one need to bring in order to answer the question: What is the next whole number after 172?

My initial thoughts mirrored another member of the group; I quickly came to the answer 173, but found it hard to say what I brought to answering the question, and I wouldn't know what to offer a student who was struggling with that question.

I recognised a different group member's suggestion that, as the question was read out, parts of the task needed to be held in the mind, pending hearing the rest of the question. I also recognised another suggestion that knowing where to separate and join sounds was important. For me, hun/dred/and/se/ven/ty/two became hundredandseventy/two.