Experiences with early algebra

Ian Benson and colleagues continue their description of children using Cuisenaire rods

In "Getting started with early algebra" we described how we have adapted the Cuisenaire-Gattegno (Cui) approach to meet the Year 1 aims for addition in the 2014 curriculum. We showed how learners can move fluently between concrete, spoken and written representations of mathematical ideas by working with colour coded rods and related symbols and notation.

Mathematical writing helps learners to recognise that they are working at the same time at four different levels. These are: the activity of putting coloured rods end to end to form a "train"; the recognition that there are many ways to choose a pair of colours to combine; the simultaneous perception of the train and its component rods; and the awareness that the written sum is both inherent in the train and distinct from it. (Gattegno, 1974, p40)

By coining the term "awareness" Gattegno gave himself the freedom to define it as a distinct movement of the mind, moment by moment, that he observed in himself and in others. "I need to start with something I can rely on," he wrote, "Is there anything more primitive than self-awareness."(Gattegno, 1986, p5)

It has been said of Gattegno's pedagogy that we can exhaustively identify the awarenesses needed in any domain, and re-define teaching as the activity which leads students "to cover this ground without missing any essential steps and without wasting time." (Young, Messum 2011, p9)

In this article we describe how we have identified steps in learning about addition, subtraction (as difference), multiplication and fractions as operators. We give examples of the learners' work after two terms in Year 1.

Equivalence, Difference, Multiplication and Fractions as Operators

Equivalence is a mathematical term that indicates that in some circumstances one thing can be substituted for another. Informally we often use the term "equals" to express this idea. For example, two trains have "equivalent length" if when placed side by side they start and end at the same place.

It may be best to first introduce this notion in a non-Cuisenaire lesson. It can be demonstrated in situations such as:

- two boys are *equivalent* in height
- two objects are *equivalent* in weight
- two balls are *unequal* in size

Then in a Cuisenaire lesson a few days later turn attention to the rods and use the language of equivalent length. For example,

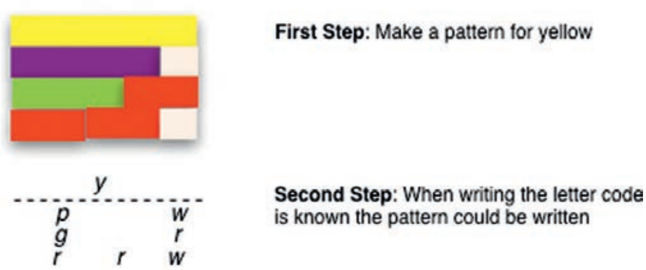
(a) Find any two rods that make a train that is *equivalent in length* to the dark green.

(b) Make a new train with these three rods and then make a train *equivalent in length* to this new one.

In this way the learner names the pattern in the illustration as "dark green plus light green" is an equivalent length to "purple plus light green plus red." It is only after such terms are used confidently in discussion and after learners have mastered the idea of using the letter codes as symbols for their colour names that the written signs should be introduced.

Introducing written symbols

Since we know that we are talking about equivalent length we can write this *equivalence* using the = sign as an equation: $d + g = p + g + r$.



y
pw
gr
rrw

Third Step. When paper train name is known the pattern could be drawn

$y \sim pw \sim gr \sim rrw$

Fourth Step: When the word equivalent is known and understood as equivalent length we can write the equivalence using the symbol ~

y
p plus w
g plus r
r plus r plus w

Fifth Step. When the term "plus" is known but not its written symbol, the pattern could be drawn

y
p + w
g + r
r + r + w

Sixth Step: When the written symbol for plus is known

$y = p + w$	$p + w = y$
$y = g + r$	$g + r = y$
$y = r + r + w$	$r + r + w = y$

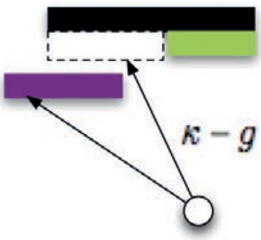
Seventh Step: Finally using the = sign we can rewrite the equation in several different forms, where *y*,*p* and *g* represent the the yellow, purple and light green lengths.

The illustration shows seven steps towards writing an equation like this using the intermediary sign ~ (for equivalent length) and Goutard's paper train names (such as *rrw*). (Goutard, 1964) The steps are based on a proposal made in Australian teacher training notes. (New South Wales, 1969)

First Step: Make a pattern for black

$k \sim pg$

Second Step: When Goutard's paper train notation is known write the name for the pattern using the equivalence sign and read it as "*k* is an equivalent length to *pg*"



$k - g$

Third Step. Demonstrate the action of removing the purple rod from the pattern and say "the pattern minus the p is called the difference between k and g"

Fourth Step: Write its name and read it as "*k* minus *g*"

First Step: Make a pattern for dark green

$d \sim 3r$

Second Step: At first dark green is equivalent to three reds

$r \sim \frac{1}{3} \times d.$

Third Step: Then one red is equivalent to a third of dark green

A similar approach is followed when introducing "minus" and unit fractions. Both involve an ambiguity that can be exploited by the teacher. In mathematics "minus" is both the name for the action of removing something and the name of a sign. Unit fractions reuse names such as "third", or "fifth". These are first encountered as the name for a step in the sequence of rods that make up a "staircase".

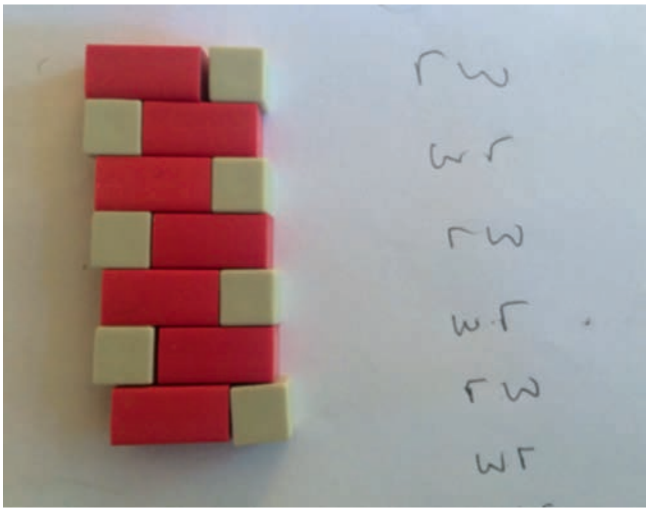
The illustration above shows how to name a pattern using the minus sign.

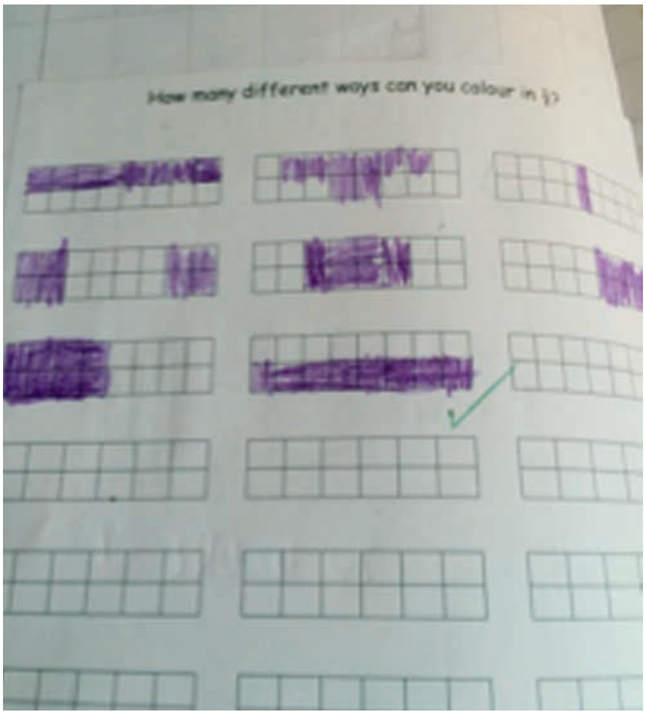
Finally, to introduce a unit fraction as an operator we first introduce a whole number as a multiplier (in 3r), and then rewrite the equation using the x sign. Re-writing was introduced in the earlier article as an action that affects the equation itself. It requires an awareness that the writing is inherent in the train and distinct from it. Re-writing gives rise to an equivalent expression or equation. If the second step is visualised as measuring the dark green with a red rod, the third step can be modelled as its inverse – measuring the red with the dark green.

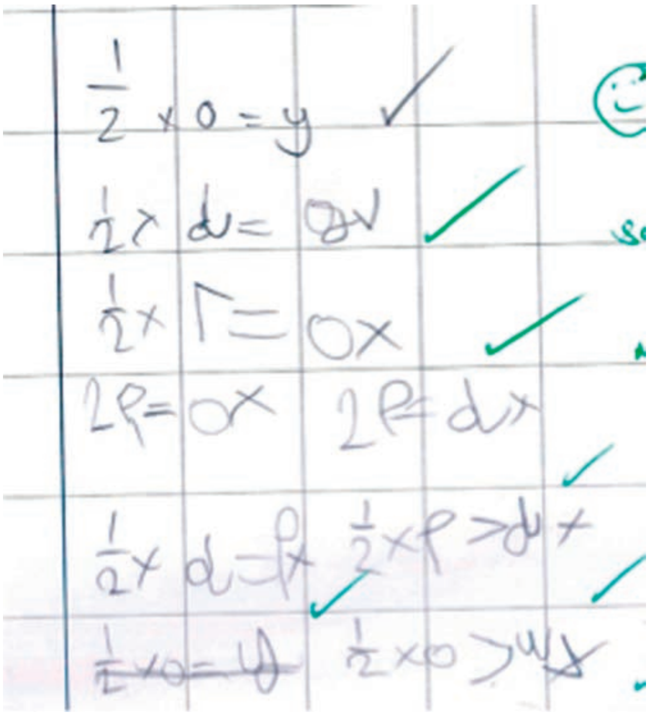
Impact of the approach

The Gattegno approach takes time, and relies on learners having frequent opportunities for free play with the rods, and free writing of equations. In a major study William Brownell, sometime Dean of Education at the University of California, Berkeley showed that after four terms "children identified as low in intelligence and exposed to a relatively long period of instruction in arithmetic will gain more from involvement in the Cui programme than in the traditional system". (Brownell, 1967)

The illustrations show how learners start by making patterns and naming the trains using





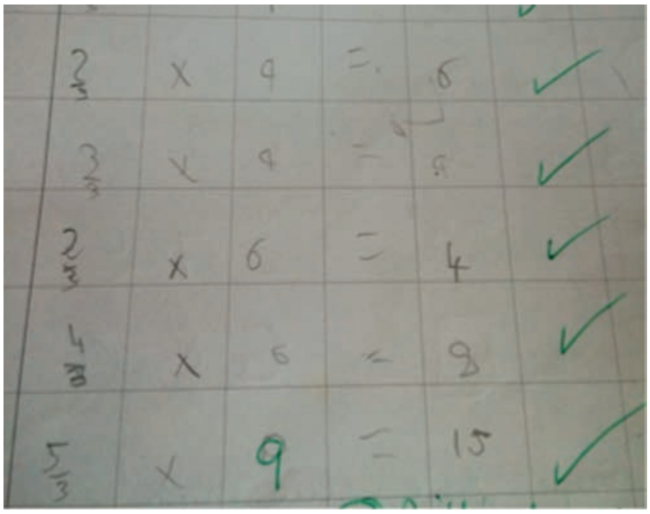


Goutard's notation. They progress to modelling plant growth using a geometric progression.

Then having mastered the notation for unit fractions learners are able to generalise to colouring fractional parts such as a half of a rectangle. They use what they know about fractions as operators to test whether equations they are given with unit fractions are true (tick) or false (cross).

The rods are then given their numeral names by measuring with the white and the learners proceed to mastery of arithmetic equations with non-unit fractions.

Conclusion



The 2014 Curriculum is a bold departure from the old National Curriculum which introduced the four operations, and fractions as operators followed by algebra in successive years. By teaching the operations together, within a progression that links concrete operations and virtual actions with images with speaking and writing, Gattegno showed how learners could master the primary curriculum in two years. Our two articles show one way in which his approach can be adapted to the new statutory entitlements and Programmes of Study.

Readers interested in learning more will find links to virtual Cuisenaire rods and Gattegno's textbook for children, "Mathematics with Numbers in Colour" at **sociality.net/imagery**.

References

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