Functional relationships between patterns of Cuisenaire rods

Ian Benson suggests how appropriate new conventions can support generalization

The study of permutations and combinations of Cuisenaire rods has proved to be a rich source of mathematical tasks motivating abstraction through algebraic symbol systems as well as mathematical generalisation. (Gattegno, 1963a, Ainsworth 2011, Benson 2011, 2014) Cuisenaire-Gattegno is a unique approach that can meet the aim set by the new national curriculum, that learners *"need to be able to move fluently between representations of mathematical ideas."* (DfE, p99)

In this exercise I take a Key Stage 3 rod permutation problem posed on the nrich site as http://nrich.maths.org/4338 (go to www.atm.org.uk/mt245 for link.)

I show how teachers can use this task to support generalisation by employing a new formalism and diagramming convention that records the functional relationships between patterns of Cuisenaire rods. This convention was suggested by William Lawvere and Stephen Schanuel as an *"external and internal diagram"* for mappings between typed sets. (Lawvere, Schanuel 1997, 2004).

The exercise requires the learner to construct specific patterns for trains equivalent in length to each rod in turn. They are restricted to using only white or red rods.

Figure 1 shows a solution to this exercise for the pink rod.

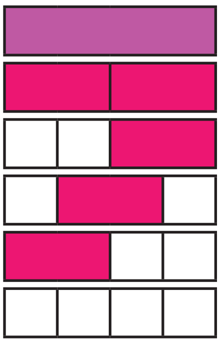

Figure 1. The pattern of white and red trains for the pink rod

Learners are asked to construct such a pattern interactively for the smaller rods, light green, pink, and yellow. Then they are asked to hypothesise how many trains would be in the pattern for the next longest - dark green - and check their work online. Finally learners are asked to say how many trains are in the pattern for orange, the longest rod, and to explain their work.

The solution is a Fibonnaci series. The new formalism enables the learner to demonstrate succinctly why this is the case, in particular why there are 89 trains in the pattern for orange.

Gattegno held strong views on how to communicate mathematically (1973, p. 2). He wrote that mathematical activity unfolds as:

1. ACTION: using the number array, the set of fingers, rods ...
2. VIRTUAL ACTION: using imagery generated by the action
3. SPEAKING: language to describe imagery
4. WRITING: symbols and notation

At first he argued that this meant that pupils didn't need text- books at all.

"Many people know that from 1953 to 1956 I refused to follow any suggestion from teachers that I should write text-books. My reasons were:

1. *that I wanted the teachers to use the rods according to their lights and not to mine*
2. *that, just as I did not want to interfere with the learning process of children, so I did not wish to interfere with the teacher's freedom of work."*
   (Gattegno, 1963b, p. 107)

However, he continued to be pressed by teachers, and eventually found the solution in the form of a log, recording a set of lessons with actual children, *"A set of questions – what better basis could be put into a book?"*. This preserved the experimental basis of the approach, as there would be no need to give answers, leaving teachers and pupils free to explore. He hoped that everyone coming to the new text-books would know that they were to be used with Cuisenaire rods, which he colour-coded in order of size w for white; r for red; g for light green; p for pink; y for yellow etc. Without the rods his books, like this note, are meaningless, *"but they become child's play when used in conjunction with the rods."* (op. cit., p. 108)

The next decision that Gattegno had to make was whether to include diagrams in his books. He chose not to do this in the books written for children between 1956 and 1960. He wrote,

… *"anyone who inserts diagrams has in my opinion*

1. *completely misunderstood the components of learning with the rods, which is a mathematisation of actions;*
2. *mixed an old-fashioned and no longer justified approach, based on images suggested by illustrations with a dynamic approach based on actions with objects, thus reducing the efficiency of both;*
3. *slowed down the learning process by spending time on irrelevant work;*
4. *fostered habits of thought which are hybrid, thus under-cutting the integrity of the learners mind."*
   (op. cit., p. 109)

By 1973 in '*Common Sense of Teaching Mathematics*' Gattegno had relented and he includes as an illustration a diagram of a set of complete patterns that he uses to define what he means by an integer. He writes, *"the set of all ways of making a particular length with rods produces an equivalence class: all trains in the set are equivalent since they have the same length*" (1974, p. 51). Figure 2 reproduces his drawing, with annotations to show the correspondences they suggest.

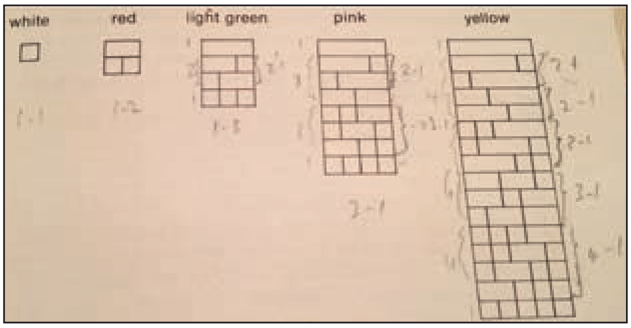

Figure 2. A set of complete patterns of rods from white to yellow

The diagram arranges the complete patterns into product groups, that is, groups of trains with the same number of cars. Looking at the number of trains in each group we see a pattern emerge corresponding to the coefficients in Newton's Binomial expansion or the sequence known conventionally as Pascal's triangle: 1, 1-1, 1-2-1, 1-3-3-1 etc.

The *nrich* exercise studies red and white trains only. From Figure 2 we can see that these patterns can be represented from left to right as sets of trains using Gattegno's coding scheme for the rods, choosing a capital letter to name each pattern.

W = { w }

G = { rw, wr, www }

R = { r, ww }

P = { rr, wwr, wrw, rww,wwww }

G and each subsequent pattern can be related to the earlier patterns by drawing a line between the members of each set as shown in Figure 3. This is called an internal diagram. An arrowhead is used to indicate the direction of the functional mapping between the sets. In this case the functions depend on the first car of the source train. In one direction trains starting with a white car map onto the previous set, and trains starting with a red car map onto the set two sets earlier. These maps are called surjections. They are shown with a closed arrowhead.

In the other direction the functions prepend a white (red) car to the source train to form the target train. This is shown with the open arrowhead. These functions are injections, or 1-to-1 mappings.

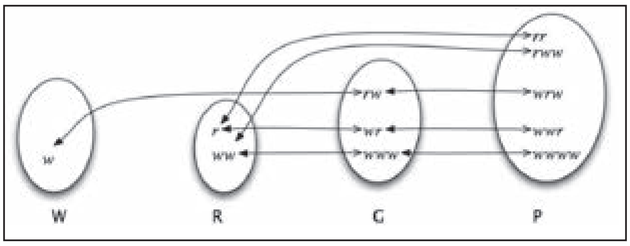

Figure 3. Functional relationships between the patterns (internal diagram)

A study of the sequence in Figure 3 shows that after G each subsequent pattern can be mapped onto the disjoint sum of the previous two sets. The number of members in each set is therefore the sum of the number of members of these prior sets.

Figure 4 is an external diagram. It extends our series to the remaining patterns (where K, N and E code for the patterns for the black, brown and blue rods). Arrows again indicate the functional relationships between the patterns.

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By inspection the number of members in the pattern for orange is 89.

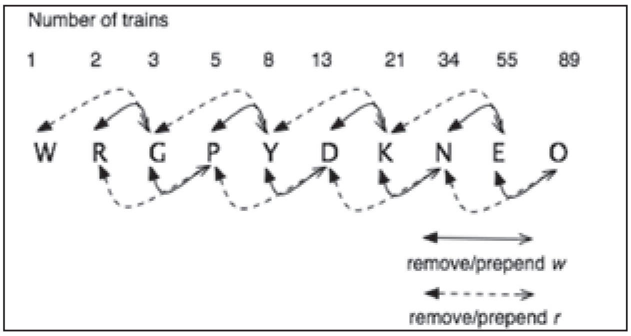

Figure 4. Functional relationships between the patterns (external diagram)

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Note Key Stage 3 rod permutation problem can be accessed at http://nrich.maths.org/4338

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