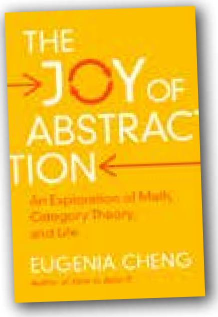

The Joy of Abstraction: An exploration of math, category theory and life by Eugenia Cheng. Published by Cambridge University Press. ISBN: 9781108769389

Reviewed by Ian Benson

One approach to mathematics is to regard it as comprising distinct aspects: algebra - the manipulation of symbols, geometry - dealing with shape and position, and logic - making arguments. Conceptual mathematics, or category theory, combines all of these. It is about the structure of arguments, and deals with algebra geometrically. While category theory only succeeds in making small portions of mathematics easy — these are the portions that lie closest to the core of the subject, the part that illuminates the rest. It lies at the heart of a revolution in how pure mathematicians do research. Since the turn of the century it has developed direct applications in ecological diversity, engineering, biology, chemistry, theoretical physics, computer science and informatics. Cheung argues that, by forcing clear thinking about contentious issues, such as social justice and gender, it can contribute to a more constructive political culture.

This book picks up from where Cheung's "Cakes, Custard and Category theory: easy recipes for understanding complex mathematics" leaves off. Cheung has the unusual talent, like Lancelot Hogben who wrote "Mathematics for the Millions," of bringing powerful ideas to the general reader. Her approach is conversational - with 200 things for the reader to stop and think about. Often these are rather vague, because part of the discipline of category theory is making precise sense of vague ideas. Taken together her two books achieve their aim of giving a non-technical audience a clear account of what it means to do technical mathematics.

In the first quarter of the book she carefully builds up the necessary intuitions: maths as patterns, relationships and equivalences, and as categories - data, structure and properties. She then devotes a short interlude to revisiting these motivating examples from the perspective of category theory: number systems, ordered sets, groups, sets and functions. Where most popular mathematics books fall shy of formalism she gently introduces formal notation, language and diagramming conventions. She argues that mathematics really does depend on this formality and makes the case that we need it in order to get further into category theory than just skimming the surface.

She investigates the relationships between numbers by multiplication, by using diagrams of arrows to help visualise structure. The formalism of drawing these diagrams help to clarify the different interactions between things. Category theory is about understanding structural reasons behind things, not just constructing formal justifications. Some proofs in mathematics just proceed step by step and arrive at a the conclusion by stringing all those steps together. She argues that a truly categorical proof does more than that, it uncovers some deep aspect of why something is happening, structurally.

One example she considers is the category of factors of 30. You might take a moment to think about what they are and how they are related? There are 8 objects {1,2,3,5,6,10,15,30}. If we draw an arrow between objects which are factors and ignore identity arrows (from an object to itself), and arrows formed by composition (such as from 2 to 30 via 6) then we have 12 arrows {1->2,1->3,1->5,2->6,2->10,3->6,3->15,5->10,5->15,6->30,10->30,15->30}. Can you draw this diagram in the form a lattice and deform it into the shape of a cube: object names labelling the vertices and the arrows labelling the edges?

Do the same for the category of factors of 42. Is this a similar diagram? In fact the two categories are "isomorphic" in several ways:

1. Each object in the category of factors of 30 has a corresponding object in the category of factors of 42, and each arrow has a corresponding arrow. Put formally we can construct a "functor" (a function that takes objects to objects and also takes arrows to arrows) and show that it is a bijection (that is one-to-one and onto) both on objects and arrows.
2. We can construct this functor from one category to the other and an inverse for it.
3. We can show that each of the two given categories is a member of the category of factors of \ital{abc} when \ital{a,b,c} are distinct primes.
4. We can show that each one is in the product category I^3 where I is the quintessential category with two objects and one arrow.

In the second 240 pages of her book she gives an account of category theory per se, such as might be found in a 100 page undergraduate text book. By taking more time she is able to put her own experience as a learner to good effect. She anticipates "gotcha's" that are regularly skipped over in the classic treatments. One innovation, that would be welcome in mainstream treatments, is the use of animation frames (she calls them videos). These show various steps in an arrow chasing proof by successively graying out and then embolding the arrows that are the developing focus of attention.

She also introduces the reader to some of the tricks of the trade in doing mathematics research. This is sound advice for all in education. She writes "A good way to read mathematical proofs: try and do it yourself, get stuck, read someone else's proof just to the point where they reveal a step that you didn't get, then try and continue by yourself, get stuck again and iterate. And "when I am doing research I get a very strong hunch that something is true but if I run into trouble proving it then I can just end up in a big mess of doubt, and flip over to trying to prove that it's not true. One can spend a lot of time flipping backwards and forwards like that. But doubt is important."

All in all this is an important book for everyone concerned with keeping their mathematics up to date. In particular mathematicians who persevere to page 440 will enjoy seeing Cayley's theorem through the lens of Yoneda's Lemma.